MOTION OF A PARABOLIC CONTOUR ON THE SURFACE OF HEAVY IDEAL LIQUID OF FINITE DEPTH

PMM Vol. 35, №3, 1971, pp. 545-549
A. V. BELOKON' and R. A. GRUNTFEST
(Rostov-on-Don)
(Received June 2, 1969)

This paper deals with the problem of motion of a parabolic contour of the from

$$f(x) = \alpha x^2 + \beta x + \gamma \qquad (\delta = ghU^{-2} < 1) \tag{0.1}$$

on the surface of a heavy ideal liquid of finite depth. Here h is the depth of the liquid, U is the velocity of motion and g is acceleration due to gravity. The methods of [1, 2] are used to construct an asymptotic solution for both, large and small values of λ, the latter quantity being the ratio of h to l (l is the half length of the zone of contact between the contour and the liquid) (see Fig. 1). As an example, a computation of the dynamic characteristics of a moving contour is given for the specific value of $\delta = 0.5$.

1. The problem under consideration, of motion of a slightly curved parabolic contour on the surface of a heavy ideal liquid of finite depth for $\delta < 1$, can be reduced using the methods of operational calculus to the pressure determination in the zone of contact, from the following integral equation [3]:

$$\int_{-1}^{1} p(u) K\left(\frac{x-u}{\lambda}\right) du = \pi f(x), \qquad |x| \leqslant 1 \tag{1.1}$$

$$K(t) = \int_0^\infty \frac{\cos \xi t}{\xi \operatorname{cth} \xi - \delta} d\xi, \qquad t = \frac{x-u}{\lambda} \tag{1.2}$$

In the formulas (0.1) and (1.1) as well as in the following, the passage to the dimensionless quantities introduced in [3] is assumed to have been performed.

In the papers dealing with gliding of a contour along the surface of a heavy ideal liquid the detailed bibliography of which appears in the monograph [4], the position of the point B of separation of the liquid from the contour was assumed known and coinciding with the trailing edge of the contour, while the position of the point A was defined from the condition of boundedness of the velocity or the pressure at B. The study of the pressure distribution under a convex contour when the point of detachment is fixed, shows that cases are possible when a negative pressure forms near the point of detachment. This is physically impossible and leads to a shift in the position of the point of detachment. This shift is determined by the condition that pressure is positive over the whole zone of contact, and this condition is met when the requirement that the pressure at the point of detachment is bounded, is supplemented by the requirement that the derivative of the pressure is also bounded along the contour (Fig. 1). We note that the last condition that the derivative of the pressure is bounded needs only hold until the point B reaches the trailing edge of the contour.

2. Following [1] we write the expression for the kernel (1.2) when λ are large, in the form

$$K(t) = -\ln|t| + a_{30} + a_{20}|t| + a_{12}t^2\ln|t| + a_{32}t^2 + a_{22}|t|^3 + O(t^4\ln^2|t|) \quad (2.1)$$

$$a_{20} = -0.5\pi\delta, \quad a_{12} = 0{,}5\delta^2, \quad a_{22} = \frac{\pi\delta^3}{12}, \quad a_{30} = \int_0^\infty \left(\frac{1}{\xi\,\mathrm{cth}\,\xi - \delta} - \frac{1-e^{-\xi}}{\xi}\right)d\xi$$

$$a_{32} = -\frac{3}{4}\delta^2 + \frac{1}{2}\int_0^\infty \left(\xi + \delta + \delta^2\frac{1-e^{-\xi}}{\xi} - \frac{\xi^2}{\xi\,\mathrm{cth}\,\xi - \delta}\right)d\xi \quad (2.2)$$

The required pressure in the zone of contact is sought in the form

$$p(x) = \omega(x)(1-x^2)^{-1/2}, \qquad \omega(x) = \sum_{i=0}^{n}\sum_{j=0}^{[0,5n]} \omega_{ij}(x)\lambda^{-i}\ln^j\lambda \quad (2.3)$$

The functions $\omega_{ij}(x)$ can be found using the formulas (1.13) of [1] and for the function $f(x)$ given by (0.1), we have

$$\omega_{00}(x) = \pi^{-1}P + \beta x - \alpha(1-2x^2)$$
$$\omega_{10}(x) = 4\pi^{-3}Pa_{20}S_1(x) + 2\pi^{-2}\beta a_{20}[2x - \Lambda(x)] - 2\pi^{-2}a_{20}\alpha S_4(x)$$
$$\omega_{20}(x) = \pi^{-1}P\{(a_{32} + 0.8069a_{12})(1-2x^2) + 32\pi^{-4}a_{20}^2[S_2 - 0.1506]\} + \beta[(a_{32} + 0.5a_{12})x + a_{12}x(x^2 - \ln 2) - 16\pi^{-4}a_{20}^2S_7(x)] + \alpha\{0.3333(2x^4 + 1.25 - 4x^2)a_{12} + 1.333\pi^{-4}a_{20}^2[S_6(x) - 2S_4(x)]\}$$

$$\omega_{21}(x) = \pi^{-1}Pa_{12}(1-2x^2) - a_{12}\beta x$$
$$\omega_{31}(x) = -2\pi^{-3}Pa_{12}a_{20}S_4(x) + 2\pi^{-2}\beta a_{20}a_{12}[\Lambda(x) - {}^{14}/_3]$$
$$\omega_{30}(x) = \pi^{-3}P\{2.667a_{22} + 0.8889a_{12}\cdot a_{20}S_3(x) + [6a_{22}(1+2x^2) - 19.30\pi^{-4}a_{20}^3]S_1(x) + 64\pi^{-4}a_{20}^3S_5(x) + [9a_{22} + 2(a_{32} + 0.8069a_{12})a_{20}]S_4(x)\} + \pi^{-2}\beta\{a_{12}a_{20}S_8(x) + 2a_{32}a_{20}[-\Lambda(x) + {}^{14}/_3x] - 16\pi^{-4}a_{20}^3S_9(x) + 2a_{22}S_{10}(x)\} - \alpha\pi^{-2}[a_{22}S_{11}(x) + a_{20}a_{12}S_{12}(x) - 2.667\pi^{-4}a_{20}^3S_{13}(x)] \quad (2.4)$$

where

$$\Lambda(x) = (1-x^2)\ln[(1-x)/(1+x)]$$

$$S_6(x) = -1.333 - 2(x^2 - 2) + 0.5[\Lambda^2(x) - (1-x^2)^2\pi^2]$$
$$S_7(x) = -2x(0.8125 - 0.1067x^2 - 0.060x^4) + (0.7067 - 0.1467x^2 - 0.060x^4)\Lambda(x)$$
$$S_8(x) = -0.0250x - 4xS_1(x) + 2.667x^3 - (2.280 + 1.333x^2)\Lambda(x)$$
$$S_9(x) = -9.309x + 1.860x^3 + 0.4354x^5 + (3.893 - 1.092x^2 - 0.2177x^4)\Lambda(x)$$
$$S_{10}(x) = 2.667x + 2x^3 - 6xS_1(x) - (2 + x^2)\Lambda(x)$$
$$S_{11}(x) = 0.2667 + (x^2 - 2.5)S_4(x) - 3S_1(x)$$
$$S_{12}(x) = 1.067 - 1.778x^2 - S_1(x) - 1.667S_4(x) + 0.6667x^2S_4(x)$$
$$S_{13}(x) = 0.3723 + 0.01523x^2 + (0.8896 - 0.1657x^2 - 0.05714x^4)S_6(x) + (-3.146 + 1.156x^2 - 0.01905x^4)S_4(x)$$

the functions $S_i(x)$, $i = 1, 2, \ldots, 5$, are given in [1].

The constant P appearing in (2.4) can be regarded as the lift of the contour and can be obtained from the condition that the solution found satisfies the initial integral equation (1.1). This yields the following relation:

$$P\varphi_1(\lambda) + \alpha\varphi_2(\lambda) = \pi\gamma \quad (2.5)$$

$$\varphi_1(\lambda) = \ln 2\lambda(1 - a_{12}\lambda^{-2} + 0.1801a_{12}a_{20}\lambda^{-3}) + a_{30} + 0.8106a_{20}\lambda^{-1} + (a_{32} + a_{12} - 0.03287a_{20}^2)\lambda^{-2} + (1.442a_{22} - 0.2702a_{12}a_{20} - 0.1807a_{32}a_{20} - 0.02450a_{20}^3)\lambda^{-3} + O(\lambda^{-4}\ln^2\lambda)$$

$$\varphi_2(\lambda) = -1.571 + 0.2829a_{20}\lambda^{-1} - 0.7854\lambda^{-2}a_{12}\ln\lambda + (0.6337a_{12} + 0.06051a_{20}^2 + 0.7854a_{32})\lambda^{-2} - 0.1698a_{12}a_{20}\lambda^{-3}\ln\lambda + (1.630a_{22} + 0.2071a_{20}a_{12} + 0.01358a_{20}^3 + 0.1698a_{20}a_{32})\lambda^{-3} + O(\lambda^{-4}\ln^2\lambda)$$

The requirement that the pressure is bounded at the point B yields yet another relation

$$\pi^{-1}P\,\varphi_3(\lambda)+\beta\varphi_4(\lambda)+\alpha\varphi_5(\lambda)=0 \tag{2.6}$$

$$\varphi_3(\lambda)=1-4\pi^{-2}a_{20}\lambda^{-1}+a_{12}\lambda^{-2}\ln\lambda+1.333\pi^{-2}a_{12}\cdot a_{20}\lambda^{-3}\ln\lambda-(a_{32}+0.8069a_{12}+4.826\pi^{-4}a_{20}^2)\lambda^{-2}-(21.33a_{22}+1.333a_{32}a_{20}+0.1867a_{12}a_{20}+9.370\pi^{-4}a_{20}^3)\pi^{-2}\lambda^{-3}+O(\lambda^{-4}\ln^2\lambda)$$

$$\varphi_4(\lambda)=1+4\pi^{-2}a_{20}\lambda^{-1}-a_{12}\lambda^{-2}\ln\lambda+(a_{32}+0.8069a_{12}+20.67\pi^{-4}a_{20}^2)\lambda^{-2}-9.333\pi^{-2}a_{20}a_{12}\lambda^{-3}\ln\lambda+(21.33a_{22}+9.333a_{32}a_{20}-6.642a_{12}a_{20}+112.6\pi^{-4}a_{20}^3)\pi^{-2}\lambda^{-3}+O(\lambda^{-4}\ln^2\lambda)$$

$$\varphi_5(\lambda)=1+1.333\pi^{-2}a_{20}\lambda^{-1}+(2.667\pi^{-4}a_{20}^2-0.25a_{12})\lambda^{-2}-(4.267a_{22}+0.9557a_{20}a_{12}-5.842\pi^{-4}a_{20}^3)\pi^{-2}\lambda^{-3}+O(\lambda^{-4}\ln^2\lambda)$$

We now turn our attention to the condition of boundedness of the pressure derivative at the point B referred to previously and which must be satisfied together with the conditions (2.5) and (2.6). It should be noted that differentiation of the asymptotic series (2.3) is not theoretically justified, therefore the process of obtaining the derivative requires another integral equation. As we know, the solution of the integral equation (1.1) with the kernel (1.2) bounded at the end point $x=1$ has the form

$$p(x)=\Omega(x)\omega_1(x),\qquad \omega_1\in C[-1,1],\qquad \Omega(x)=(1-x)^{1/2}(1+x)^{-1/2}$$

From this it follows that the function

$$q(x)=p(x)-a\Omega(x),\qquad a=\omega_1(-1)=-\beta\varphi_4(\lambda)$$

is bounded on the interval $[-1,1]$ and vanishes at both its ends. Let us write the integral equation (1.1) in the form

$$\int_{-1}^{1}q(u)K\left(\frac{x-u}{\lambda}\right)du=\pi f(x)-a\int_{-1}^{1}\Omega(u)K\left(\frac{x-u}{\lambda}\right)du$$

Differentiation with respect to x followed by integration by parts of the left hand side yields

$$\int_{-1}^{1}\frac{dq(u)}{du}K\left(\frac{x-u}{\lambda}\right)du=\pi f'(x)-a\int_{-1}^{1}\Omega(u)K_x'\left(\frac{x-u}{\lambda}\right)du \tag{2.7}$$

For large values of λ a solution of this integral equation is constructed analogously to that of (1.1). Omitting numerous calculations we can write at once the condition of boundedness of the pressure derivative at the point B

$$4\alpha+\beta\varphi_6(\lambda)=0 \tag{2.8}$$

$$\varphi_6(\lambda)=1+8\pi^{-2}a_{20}\lambda^{-1}-4a_{12}\lambda^{-2}\ln\lambda+(4a_{32}+5.228a_{12}+51.38\pi^{-4}a_{20}^2)\lambda^{-2}-32a_{20}a_{12}\pi^{-2}\lambda^{-3}\ln\lambda-(128a_{22}-32a_{20}a_{32}-31.15a_{20}a_{12}-156.2\pi^{-4}a_{20}^3)\pi^{-2}\lambda^{-3}+O(\lambda^{-4}\ln^2\lambda)$$

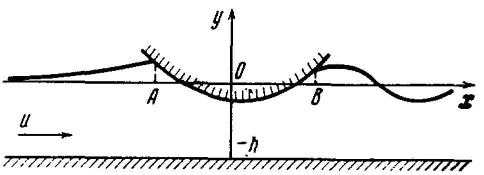

Fig. 1

We note that differentiation of the asymptotic series (2.3) yields a different result. From (2.6)–(2.8) we easily obtain

$$\frac{P^*}{\pi\alpha^*\rho U^2h^2}\varphi_3(\lambda)\varphi_6(\lambda)\lambda^2+\varphi_5(\lambda)\varphi_6(\lambda)-4\varphi_4(\lambda)=0 \tag{2.9}$$

which determines the parameter λ and consequently the length of the zone of contact for P^* (dimensional weight of the contour) and α^* (dimensional curvature) given. Having found λ we can find the parameters β and γ defining the position of the contour relative to the coordinate system used (Fig. 1) from the expressions (2.6) and (2.5). Finally we find the moment of the pressure forces M relative to the coordinate origin and the drag W experienced by the contour during the motion

$$M = \beta\varphi_7(\lambda), \qquad W = \beta P + 2\alpha M \tag{2.10}$$

$$\varphi_7(\lambda) = 1.571 + 0.8488a_{20}\lambda^{-1} - 1.571a_{12}\lambda^{-2}\ln\lambda + (0.8748a_{12} + 1.571a_{32} + 0.4017a_{20}^2)\lambda^{-2} - 1.698a_{20}a_{12}\lambda^{-3}\ln\lambda + (2.716a_{22} + 0.9170a_{12}a_{20} + 1.698a_{32}a_{20} + 26.81\pi^{-4}a_{20}^3)\lambda^{-3} + O(\lambda^{-4}\ln^2\lambda)$$

3. Following [2] we take the asymptotic solution of (1.1) for small values of λ in the following form:

$$p(x) = p_1\left(\frac{1+x}{\lambda}\right) + p_2\left(\frac{1-x}{\lambda}\right) - p_3(x) \tag{3.1}$$

or

$$p(x) = p_1\left(\frac{1+x}{\lambda}\right) p_2\left(\frac{1-x}{\lambda}\right) p_3^{-1}(x) \tag{3.2}$$

where $p_i(t)$, $i = 1, 2$ are the solutions of the Wiener-Hopf integral equations

$$\int_0^\infty p_i(u) K(t-u)\,du = \pi\lambda^{-1}(\alpha t^2 + \beta_i t + \gamma_i), \qquad 0 \leqslant t < \infty \tag{3.3}$$

$$\gamma_i = \alpha + (-1)^i\beta + \gamma, \qquad \beta_i = -2\alpha - (-1)^i\beta \qquad (i = 1, 2)$$

and $p_3(x)$ is the degenerate solution obtained when $\lambda \to 0$.

In obtaining the solutions of the integral equations (3.2) an approximation of the form

$$\frac{\sqrt{\xi^2 + B^2}}{\xi^2 + C^2}, \qquad A = \frac{B}{C^2} = \frac{1}{1-\delta} \tag{3.4}$$

was used for the Fourier transform of the kernel (1.2). This led to the following result

$$p_i(t) = \frac{1}{A}\left[\operatorname{erf}\sqrt{Bt} + \left(\frac{A}{\pi t}\right)^{1/2} e^{-Bt}\right]\left(\frac{\gamma_i}{\lambda} - \beta_i\varepsilon_1 - \frac{\alpha\lambda}{B}\varepsilon_2\right) + \frac{1}{A}\left[(t+\varepsilon_1)\operatorname{erf}\sqrt{Bt} + \left(\frac{t}{\pi B}\right)^{1/2} e^{-Bt}\right](\beta_i - 2\alpha\lambda\varepsilon_1) + \frac{2\alpha\lambda}{A}\left[\left(\frac{t^2}{2} + t\varepsilon_1 - \frac{1}{2B}\varepsilon_2\right)\operatorname{erf}\sqrt{Bt} + \left(\frac{t}{2} + \varepsilon_2\right)\left(\frac{t}{\pi B}\right)^{1/2} e^{-Bt}\right]$$

$$\varepsilon_1 = \frac{1}{C} - \frac{1}{2B}, \quad \varepsilon_2 = \frac{1}{C} - \frac{3}{4B}, \quad p_3(x) = \frac{1}{A\lambda}(\alpha x^2 + \beta x + \gamma) \tag{3.5}$$

As in [2], the solution of the form (3.2) is used to obtain the moment M of the pressure forces relative to the coordinate origin and the lift P of the contour. We have

$$M = \beta\psi_1(\lambda) \tag{3.6}$$

$$\psi_1(\lambda) = \frac{2}{A}\left[\frac{1}{3\lambda} + \varepsilon_1 + \lambda\varepsilon_1^2 + \lambda^2\left(\varepsilon_1\frac{(B-C)^2}{B^2C^2} - \varepsilon_1^3 + \frac{(B-C)^2}{2B^3C^2}\right)\right] - \exp\left(-\frac{2B}{\lambda}\right)\left[1 + 2\lambda\left(\varepsilon_1 + \frac{1}{B}\right) + \lambda^2\left(\varepsilon_1 + \frac{1}{B}\right)^2\right]\frac{(B-C)^2}{B^2}$$

$$P = \gamma\psi_2(\lambda) + \alpha\psi_3(\lambda), \quad \psi_2(\lambda) = \frac{1}{A}\left(\frac{2}{\lambda} + 2\varepsilon_1\right) + \frac{(B-C)^2}{B^2}\exp\left(\frac{-2B}{\lambda}\right) \tag{3.7}$$

$$\psi_3(\lambda) = \frac{2}{A}\left[\frac{1}{3\lambda} + \varepsilon_1 - \frac{\lambda\varepsilon_2}{B} + \lambda^2\varepsilon_3 + \left(\frac{(B-C)^2}{4BC^2} - \frac{\lambda}{2B}\left(\varepsilon_1 - \frac{1}{2B}\right) - \lambda^2\varepsilon_3\right)\exp\left(\frac{-2B}{\lambda}\right)\right], \qquad \varepsilon_3 = \frac{3}{4B^2C} - \frac{5}{8B^3}$$

The conditions that the pressure and its derivative are both bounded at the point B lead to the following relations:

$$\gamma + \beta\psi_4(\lambda) + \alpha\psi_5(\lambda) = 0, \qquad \psi_4(\lambda) = 1 + \lambda\varepsilon_1 \tag{3.8}$$
$$\psi_5(\lambda) = 1 + 2\lambda\varepsilon_1 - \lambda^2 B^{-1}\varepsilon_2, \qquad \beta + 2\alpha\psi_4(\lambda) = 0$$

The relations (3.6)–(3.8) easily yield an equation for the definition of λ which corresponds to (2.9) in the method used for large λ

$$\frac{P^*}{\rho U^2 h^2 \alpha^*} = \lambda^{-2}\left[2\psi_4^2(\lambda)\psi_2(\lambda) + \psi_3(\lambda) - \psi_5(\lambda)\psi_3(\lambda)\right] \tag{3.9}$$

With λ known, the parameters β and γ are obtained from (3.8) and (3.9), while the moment M and the drag W can be found from (3.6) and (3.2).

Numerical computations show that the formulas of the Sects.2 and 3 providing the solutions of the stated problem for large and small λ, respectively, embrace the whole interval of variation of λ $(0 < \lambda < \infty)$. Matching of the methods occurs near the point $\lambda=1.5$.

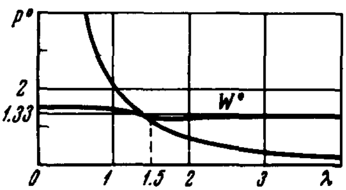

Fig. 2

As an example, we consider the formulas obtained for $\delta = 0.5$. We have

$$a_{20} = -0.25\pi, \qquad a_{12} = 0.125$$
$$a_{32} = 0.2629, \qquad a_{22} = -{}^1/_{96}\pi$$
$$B = 2.725, \qquad C = 1.362, \qquad A = 2$$

Results of the computation are given in Fig. 2 and Table 1. The methods meet near the point $\lambda = 1.5$ and the relative difference in the results at $\lambda = 1.5$ is 3% for the lift coefficient $P^\circ = P^*/\pi\rho U^2 h^2 \alpha^*$ and 1% for the drag coefficient $W^\circ = W / \beta P$.

Table 1

λ	0	0.25	0.5	1.0	1.5	2	5	10	∞
P°	∞	43.45	8.240	2.055	1.196				
					1.157	0.6854	0.1200	0.03012	0
W°	1.5	1.496	1.477	1.420	1.317				
					1.310	1.315	1.323	1.328	1.333

BIBLIOGRAPHY

1. Aleksandrov, V. M. and Belokon', A. V., Asymptotic solution of a class of integral equations and its application to contact problems for cylindrical elastic bodies. PMM Vol. 31, №4, 1967.
2. Aleksandrov, V. M. and Babeshko, V. A., Contact problems for a thin elastic strip. Izv. Akad. Nauk SSSR, Mekhanika, №2, 1965.
3. Belokon', V. M. and Gruntfest, R. A., On the problem of gliding of a plate on the surface of a heavy ideal liquid of finite depth. PMM Vol. 34, №5, 1970.
4. Sedov, L. I., Plane Problems in Hydro and Aerodynamics, ed. 2. M., "Nauka", 1966.

Translated by L. K.